Alexander Polynomials and Gordian Distance

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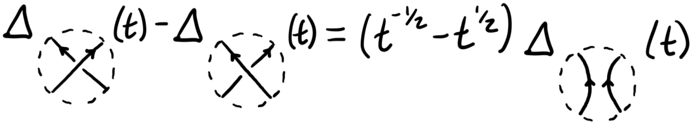
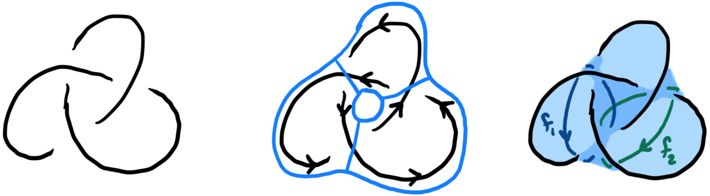
Alexander Polynomial

Computation:

- For a Seifert matrix A of a knot, $\det(tA - A^T)$ is the Alexander polynomial
- Surgery on the knot exterior
- Skein relation



Example: Trefoil Using Seifert Matrix

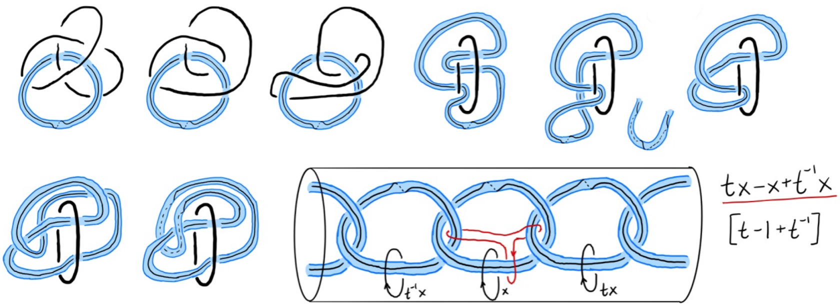


$$A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$tA - A^T = \begin{pmatrix} -t & -t \\ 0 & -t \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -t+1 & -t \\ 1 & -t+1 \end{pmatrix}$$

$$\triangle_{3_1}(t) = \det(tA - A^T) = t^2 - 2t + 1 + t = t^2 - t + 1$$

Example: Trefoil Using Surgery



$$\triangle_{3_1}(t) = \det[t - 1 + t^{-1}] = t - 1 + t^{-1}$$

Example: Trefoil Using Skein Relation

$$\Delta(t) - \Delta(t) = (t^{-1/2} - t^{1/2})\Delta(t)$$

$$\Delta_{3_1}(t) - 1 = (t^{-1/2} - t^{1/2})\Delta(t)$$

$$\Delta(t) - \Delta(t) = (t^{-1/2} - t^{1/2})\Delta_{\circ}(t)$$

$$\Delta(t) - \Delta(t) = (t^{-1/2} - t^{1/2})\Delta(t)$$

$$\Delta_{\circ\circ}(t) = 0$$

$$\Delta(t) = t^{-1/2} - t^{1/2}$$

$$\triangle_{3_1}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 + 1 = t^{-1} - 2 + t + 1 = t - 1 + t^{-1}$$

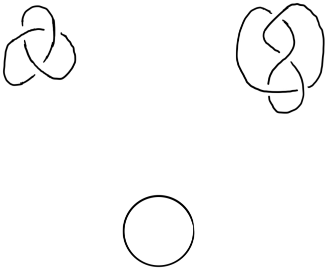
Alexander Polynomial

Definition: Given a knot K, we can view $H_1(X_\infty)$ as a module over $\mathbb{Z}[t, t^{-1}]$ where X_∞ is the infinite cyclic cover of the complement of K in S^3. The Alexander polynomial $\triangle_K(t)$ of K is the determinant of a presentation matrix (called an Alexander matrix) of $H_1(X_\infty)$ as a module over $\mathbb{Z}[t, t^{-1}]$. This is unique up to multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$ $(\pm t^n)$.

Characterization: For any oriented knot K, $\triangle_K(t) = \triangle_K(t^{-1})$ up to multiplication by a unit and $\triangle_K(1) = \pm 1$.

Gordian Distance

Definition: The Gordian distance between two knots is the minimum number of crossing changes required to change from one to the other.



The Gordian distance between any knot and the unknot is its unknotting number.

Alexander Polynomial and Unknotting Number 1

Theorem (Kondo, 1978): For any Alexander polynomial p(t), there exists a knot K with unknotting number 1 such that $\triangle_K(t) = p(t)$.

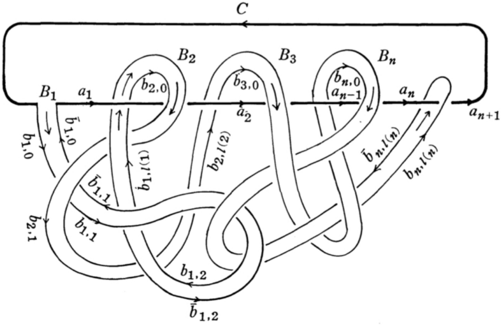


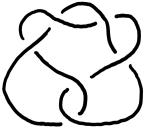
Fig. 4

Interaction Between Alexander Polynomial and Gordian Distance

Question: Does there exist a nontrivial Alexander polynomial $a(t)$ such that for any Alexander polynomial $b(t)$, there exist a pair of knots K_a and K_b with Gordian distance 1 such that $\triangle_{K_a} = a(t)$ and $\triangle_{K_b} = b(t)$?

Answer (Kawauchi, 2011): Yes! This is the case for any Alexander polynomial $a(t)$ of slice type (meaning that $a(t) = c(t)c(t^{-1})$ for some Laurent polynomial $c(t)$.)

Example: $\triangle_{6_1}(t) = -2t + 5 - 2t^{-1} = (-2t + 1)(-2t^{-1} + 1)$

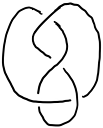


Interaction Between Alexander Polynomial and Gordian Distance

Jong's Problem: Does there exist a pair of Alexander polynomials $a(t)$ and $b(t)$ such that any two knots K_a and K_b where $\triangle_{K_a} = a(t)$ and $\triangle_{K_b} = b(t)$ have Gordian distance at least 2?

Answer (Kawauchi, 2011): Yes! For example

$$a(t) = t - 1 + t^{-1}$$
$$b(t) = -t + 3 - t^{-1}$$



Interaction Between Alexander Polynomial and Gordian Distance

Question: Does there exist a knot K with nontrivial Alexander polynomial such that for any Alexander polynomial $a(t)$, there exists some knot K_a such that $\triangle_{K_a}(t) = a(t)$ and the Gordian distance between K and K_a is 1?

Answer: Open

Characterizing the Alexander polynomials of knots Gordian distance 1 from K

Notation: For any knot K, let $K^\times$ be the knot Gordian distance 1 from K and let $\triangle K^\times$ be the set of Alexander polynomials of knots in $K^\times$. Let $\triangle\mathcal{K}$ be the set of all Alexander polynomials.

Theorem (Nakanishi and Okada, 2011):

$$\begin{aligned}
\triangle 10_{132}^\times \cap \triangle 5_1^\times &\neq \emptyset \\
\triangle 10_{132}^\times \setminus \triangle 5_1^\times &\neq \emptyset \\
\triangle 5_1^\times \setminus \triangle 10_{132}^\times &\neq \emptyset \\
\triangle\mathcal{K} \setminus (\triangle 10_{132}^\times \cup \triangle 5_1^\times) &\neq \emptyset
\end{aligned}$$

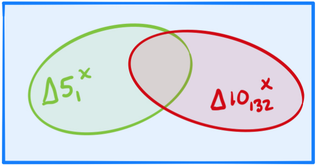
Characterizing the Alexander polynomials of knots Gordian distance 1 from K

5_1

10_{132}

$$\triangle_{5_1}(t) = t^2 - t + 1 - t^{-1} + t^{-2} = \triangle_{10_{132}}(t)$$

ΔK



What knots have been eliminated from the open question?

References

1. Kawauchi, Akio: ”On the Alexander polynomials of knots with Gordian distance one” 2011
2. Kondo, Hisako: ”Knots of Unknotting Number 1 and Their Alexander Polynomials” 1978
3. Nakanishi, Yasutaka & Okada, Yuki: ”Differences of Alexander polynomials for knots caused by a single crossing change” 2011